

A CURVE COMPLEX AND INCOMPRESSIBLE SURFACES IN $S \times \mathbb{R}$

INGRID IRMER

ABSTRACT. Various curve complexes with vertices representing multicurves on a surface S have been defined, for example [2], [3] and [6]. The homology curve complex $\mathcal{HC}(S, \alpha)$ defined in [5] is one such complex, with vertices corresponding to multicurves in a nontrivial integral homology class α . Given two multicurves m_1 and m_2 corresponding to vertices in $\mathcal{HC}(S, \alpha)$, it was shown in [6] that a path in $\mathcal{HC}(S, \alpha)$ connecting these vertices represents a surface in $S \times \mathbb{R}$, and a simple algorithm for constructing minimal genus surfaces of this type was obtained. In this paper, a Morse theoretic argument will be used to prove that all embedded orientable incompressible surfaces in $S \times \mathbb{R}$ with boundary curves homotopic to $m_2 - m_1$ are homotopic to a surface constructed in this way. This is used to relate distance between two vertices in $\mathcal{HC}(S, \alpha)$ to the Seifert genus of the corresponding link in $S \times \mathbb{R}$.

1. INTRODUCTION

Suppose S is a closed oriented surface, each connected component of which has genus at least 2.

Let π be the projection of $S \times \mathbb{R}$ onto $S \times 0$ given by $(s, r) \mapsto (s, 0)$. A *multicurve* in $S \times \mathbb{R}$ is a one dimensional embedded submanifold that projects onto a one dimensional embedded submanifold of $S \times 0$. It is also assumed that multicurves do not contain curves that bound discs.

Fix a nontrivial element α of $H_1(S, \mathbb{Z})$. The *homology curve complex*, $\mathcal{HC}(S, \alpha)$, is a simplicial complex whose vertex set is the set of all isotopy classes of oriented multicurves in S in the homology class α . A set of vertices $m_1 \dots m_k$ spans a simplex if the representatives of the isotopy classes can all be chosen to be disjoint.

As described in [6], a path in $\mathcal{HC}(S, \alpha)$ corresponds to a surface in $S \times \mathbb{R}$. This construction will be briefly repeated in section 2. The main theorem of this paper, theorem 1.1, shows a converse of this.

All homotopies of surfaces in $S \times \mathbb{R}$ are assumed to be smooth, and are allowed to move the boundaries of surfaces.

Let m_1 and m_2 be homologous multicurves in $S \times \mathbb{R}$ representing vertices $\mathcal{HC}(S, \alpha)$.

Theorem 1.1. *Suppose H is an oriented, embedded, incompressible surface in $S \times \mathbb{R}$ with boundary $m_2 - m_1$. Then there exists a path γ in $\mathcal{HC}(S, \alpha)$ connecting the vertices corresponding to m_1 and m_2 such that H is homotopic to a surface constructed from γ .*

The basic idea of the proof is to use the coordinate obtained by projecting the surface onto the second component of $S \times \mathbb{R}$ as a Morse function m_R on the surface. This is shown to be possible in lemma 3.1.

An edge in $\mathcal{HC}(S, \alpha)$ connecting two vertices representing the multicurves γ_i and γ_{i+1} is called *simple* if $\gamma_{i+1} - \gamma_i$ is the oriented boundary of an embedded subsurface of S . The convention is that if γ_{i+1} and γ_i both contain a curve c , the submulticurve $c - c$ of $\gamma_{i+1} - \gamma_i$ bounds an annulus, not the empty set. It then follows that a *simple path*, i.e. a path that only passes through simple edges, gives rise to an embedded surface in $S \times \mathbb{R}$.

The geometric intersection number, $i(m_1, m_2)$, of two multicurves in $S \times \mathbb{R}$ is defined by projecting onto $S \times 0$. Recall that the geometric intersection number of two multicurves m_1 and m_2 is the minimum possible number of intersections between a pair of multicurves, one of which is isotopic to m_1 and the other to m_2 .

The *distance*, $d_{\mathcal{H}}(v_1, v_2)$, between two vertices v_1 and v_2 in $\mathcal{HC}(S, \alpha)$ is defined to be the distance in the path metric of the one-skeleton, where all edges have length one.

In [6] it was shown that the smallest possible genus of a surface in $S \times \mathbb{R}$ with boundary homotopic to $m_2 - m_1$ provides a bound from below on distance in $\mathcal{HC}(S, \alpha)$ between two vertices represented by multicurves m_1 and m_2 . Theorem 1.1 shows the converse, namely, the smallest possible genus of a surface in $S \times \mathbb{R}$ with boundary curves homotopic to $m_2 - m_1$ gives a bound from above on the distance in $\mathcal{HC}(S, \alpha)$ between m_1 and m_2 . A corollary of theorem 1.1 is used in [6] to obtain a simple, $\mathcal{O}(i(m_1, m_2))$ algorithm for constructing minimal genus surfaces in $S \times \mathbb{R}$ with boundary $m_2 - m_1$. This is in contrast to the problem of finding a minimal genus surface of a knot embedded in a general 3-manifold, which was shown in [1] to be **NP**-complete.

Corollary 1.2 (Distance in $\mathcal{HC}(S, \alpha)$ and genus of surfaces). *Let $d_{\mathcal{C}}(m_1, m_2)$ be the distance in $\mathcal{HC}(S, \alpha)$ between the vertices corresponding to the multicurves m_1 and m_2 , and g_H be the smallest possible genus of a surface in $S \times \mathbb{R}$ with boundary $m_2 - m_1$. Then there exist constants k_1 and k_2 depending only on the genus of S such that*

$$k_1 g_H \leq d_{\mathcal{C}}(m_1, m_2) \leq k_2 g_H$$

Acknowledgements. I would like to thank Ursula Hamenstädt for her supervision of this project, and Dan Margalit for his advice.

2. CONSTRUCTING SURFACES FROM PATHS IN $\mathcal{HC}(S, \alpha)$

Whenever this does not lead to confusion, the same symbol will be used for a vertex in $\mathcal{HC}(S, \alpha)$ and the corresponding multicurve on S . Also, a path in $\mathcal{HC}(S, \alpha)$ will often be denoted by a sequence of multicurves, m_1, m_2, \dots, m_n with the property that m_i and m_{i+1} are disjoint for every $1 \leq i \leq n$, i.e. m_i and m_{i+1} represent an edge in $\mathcal{HC}(S, \alpha)$.

Let $\gamma := \{\gamma_0, \gamma_1, \dots, \gamma_j\}$ be the vertices of a simple path in $\mathcal{HC}(S, \alpha)$. Since S maps into $S \times \mathbb{R}$, each γ_i represents a multicurve in $S \times \mathbb{R}$.

A surface T_γ contained in $S \times j \subset S \times \mathbb{R}$ is constructed inductively. Given γ_0 , isotope γ_1 such that there is a subsurface S_1 of S with boundary $\gamma_1 - \gamma_0$. Let T_1 be the surface in $S \times [0, 1]$ given by $\gamma_0 \times [0, \frac{1}{2}] \cup S_1 \times \{\frac{1}{2}\} \cup \gamma_1 \times [\frac{1}{2}, 1]$. Next, isotope γ_2 so that there is a subsurface S_2 of S with $\partial S_2 = \gamma_2 - \gamma_1$ and let $T_2 = \gamma_1 \times [1, \frac{3}{2}] \cup S_2 \times \{\frac{3}{2}\} \cup \gamma_2 \times [\frac{3}{2}, 2]$. Repeat this successively for each of the γ_i until an embedded surface $T_\gamma = T_1 \cup T_2 \cup \dots \cup T_j$ in $S \times [0, j]$ is obtained.

When constructing a surface in $S \times \mathbb{R}$ from a path γ in $\mathcal{HC}(S, \alpha)$, at every step there is a choice involved. It is possible to attach S_i , as in the previous construction, or its complement in $S \times 0$ with the opposite orientation. Call all such surfaces *surfaces constructed from γ* . The path γ behaves like a partial marking of the surface, and is not in general unique.

3. MORSE THEORY WITH BOUNDARY

All multicurves and manifolds are assumed to be smooth throughout this paper. The manifold $S \times \mathbb{R}$ is given a product metric $ds_M^2 = ds_S^2 + dR^2$ where ds_S is a choice of metric on $S \times 0$. Similarly, H and all surfaces in $S \times \mathbb{R}$ homotopic to H are assumed to be covered by coordinate charts $(U_1, s_1, R), \dots, (U_k, s_k, R)$, where the s_i are coordinates obtained by projecting onto $S \times 0$.

Whenever H is embedded in $S \times \mathbb{R}$, it is a corollary of theorem 1.1 that there is a homotopy of H that takes H to an embedded surface in $S \times [a, b]$ with boundary contained in the level sets $S \times a$ and $S \times b$. However, this is not something that can be assumed a priori. In order to deal with the possibility that all homotopies of H that put ∂H into two level sets might introduce self-intersections of H , the standard Morse theory has to be modified slightly.

A *critical point* of a Morse function on ∂H is a point where the restriction of the Morse function to the boundary has zero derivative, and a *degenerate critical point* on the boundary is any critical point that is not an isolated local extremum.

Lemma 3.1. *Suppose H is a compact embedded surface in M with boundary consisting of the multicurves m_1 and m_2 . Then there is an embedded surface in $S \times \mathbb{R}$, call it H' , with the following properties:*

- (1) H' is homotopic to H
- (2) The restriction, m_R , of the \mathbb{R} coordinate to H' is a Morse function
- (3) No two critical points of the Morse function from 2 have the same value of the \mathbb{R} coordinate.

Proof. It is a standard result, e.g. [7] Theorem 2.7, that on a compact manifold without boundary, the Morse functions form an open, dense (in the \mathcal{C}^2 topology) subset of the set of all smooth functions of the manifold into \mathbb{R} . This and similar standard results in Morse theory are proven by altering a given function by adding arbitrarily small functions with small derivatives. Similar arguments are used here; the main difference is that the coordinate R is treated as fixed while the subset of $S \times \mathbb{R}$ to which R is restricted is altered by a homotopy.

In the proof that the Morse functions form an open dense subset of H into \mathbb{R} , first of all the existence of a surface H^1 homotopic to H on which m_R is a Morse function on some neighbourhood of the boundary will be shown. The standard Morse theory arguments (e.g. theorem 2.7 of [7]) that assume empty boundary then apply to H^1 , from which claims 2 and 3 of the lemma follow. It will then be shown that if the homotopies representing these alterations are sufficiently close to the identity, embeddedness is preserved.

Existence of a homotopy of H that makes m_R Morse on some neighbourhood of the boundary. Let N be a collar of the boundary of H ; the existence of which is guaranteed by theorem 6.1, chapter 4 of [4]. The boundary of H is a compact manifold without boundary, so by theorem 2.7 of [7], if the restriction of R to ∂H is not a Morse function, there is a Morse function R_m on ∂H arbitrarily close to R in the \mathcal{C}^2 topology.

The collar N is diffeomorphic to several copies of $S^1 \times [0, \iota]$, which defines coordinates (t, r) on each component of N , where t is the parameter on S^1 and r is defined on the interval $[0, \iota]$ and is equal to zero on the boundary curves m_1 and m_2 . Let $\phi(t, r)$ be a smooth function on N , $0 \leq \phi \leq 1$, $\phi(t, 0) = 1$, and let $\eta(t)$ be the function $R_m(t) - R$ on

∂H . It follows that $R + \phi(t, r)\eta(t)$ is a Morse function when restricted to ∂H , i.e. for $r = 0$. To construct a function without degenerate critical points on a neighbourhood of the boundary, it is enough to show that $\phi(t, r)$ can be chosen such that $\frac{d(R+\phi(t,r)\eta(t))}{dr}$ and $\frac{d(R+\phi(t,r)\eta(t))}{dt}$ are not simultaneously zero on a neighbourhood N_1 of ∂H contained in N .

As a consequence of smoothness, $\frac{d(R+\phi(t,\kappa)\eta(t))}{dt} - \frac{d(R+\phi(t,0)\eta(t))}{dt}$ can be made arbitrarily small by choosing κ sufficiently small. Since $R + \phi(t, r)\eta(t)$ is a Morse function on ∂H , when restricted to ∂H , $\frac{d(R+\phi(t,r)\eta(t))}{dt}$ is only zero at (isolated) critical points $p_1 = (t_1, 0)$, $p_2 = (t_2, 0)$, \dots , $p_n = (t_n, 0)$. Therefore, $N_1 \subset N$ can be chosen such that in N_1 , $\frac{d(R+\phi(r,t)\eta(t))}{dt}$ can only pass through zero in a neighbourhood of the form $P_i := (p_i - \epsilon, p_i + \epsilon) \times (0, \epsilon)$, for $i = 1, 2, \dots, n$. Inside each of the P_i , ϕ can be chosen such that $\frac{d(R+\phi(r,t)\eta(t))}{dr}$ is nonzero. This is possible because ϵ can be chosen such that R , η and their derivatives do not vary much in the ϵ neighbourhoods. It follows that N_1 and ϕ can be chosen such that $R + \phi(r, t)\eta(t)$ is a Morse function on N_1 .

Preserving embeddedness It remains to show that when the homotopy taking H to H' is chosen to be sufficiently close to the identity, embeddedness is preserved. Let H^1 be a (possibly immersed) surface with boundary in $S \times \mathbb{R}$ that coincides with H outside of N and is given by the graph $(s, R + \phi(r, t)\eta(t))$ in the coordinate chart (U_i, s_i, R) over N . Since H is smoothly embedded in $S \times \mathbb{R}$ as a submanifold with boundary, it follows from theorems 6.1 and 6.3 of [4] that H has an embedded neighbourhood $\mathcal{E}(H)$ in $S \times \mathbb{R}$. As R_m approaches R in the \mathcal{C}^2 topology on ∂H , $R + \phi(r, t)\eta(t)$ also approaches R in the \mathcal{C}^2 topology on N . If R_m was chosen to be sufficiently close to R in the \mathcal{C}^2 topology, it follows that H^1 is contained in $\mathcal{E}(H)$ and is also embedded.

Setting $H' = H^1$ for R_m sufficiently close to R therefore gives a surface with the properties claimed in the statement of the lemma. \square

4. ORDINARY HANDLES AND BOW TIE HANDLES

In the proof of theorem 1.1 it is necessary to keep track of intersection properties of projections of curves and arcs to $S \times 0$. For this reason it is helpful to distinguish between two distinct methods of attaching handles, depending on the way the handle projects into $S \times 0$.

Let $H_b^a := H \cap (S \times [b, a])$, $H_b := H \cap (S \times [b, \infty))$, $H^a := H \cap ((-\infty, a])$ and $H(a) := H \cap (S \times a)$.

4.1. Ordinary handles. Suppose H^a contains two components, F_1 and F_2 , that are subsurfaces of a connected component F of $H^{a+\delta}$ for

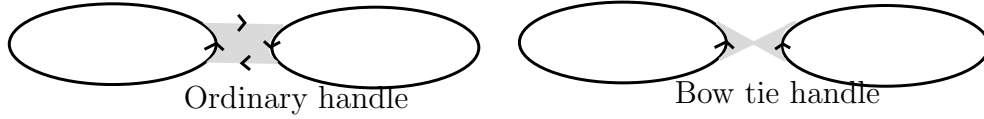


FIGURE 1. Examples of the two different types of handles.

small δ . In other words, there is a component of $H^{a+\delta}$ obtained by attaching a handle to $F_1 \cup F_2$. In two dimensions, a handle can be thought of as an oriented rectangle Q in $S \times \mathbb{R}$, whose boundary is a union of four arcs, each given an orientation as a subarc of the boundary of Q . A pair of opposite sides of Q , q^1 and q^2 , are glued along arcs on the boundary components of F_1 and F_2 respectively, in such a way that pairs of arcs with opposite orientation are glued together. In this way, an oriented surface F is obtained, such that F_1 and F_2 are oriented as subsurfaces of F .

4.2. Bow tie handles. Whenever F_1 and F_2 project onto subsurfaces of $S \times 0$ with opposite orientations, i.e. $\pi(F_1)$ is oriented as a subsurface of $S \times 0$, and $\pi(F_2)$ has the opposite orientation, the handle Q has to be embedded in $S \times \mathbb{R}$ with an odd number of half twists, otherwise the orientations of F_1 and F_2 can not be made to match up. The aim is to make a definition to distinguish between “ordinary” handles and those that contain twists, which will be called “bow tie” handles. There are two complications to doing this:

- (1) A rectangle Q as in the previous example contains a rectangle without twists; the representative of the homotopy class of $H^{a+\delta}$ might be chosen such that this rectangle in Q could be viewed as an “ordinary” handle connecting two components of H^a .
- (2) An “ordinary” handle, when given a half twist in one direction and a half twist in the other direction to cancel it out, could be viewed as two handles with twists.

In order to avoid complications 1 and 2, when determining whether or not a handle contains twists, a representative of the homotopy class of H^a is chosen that avoids all nonessential points of intersection of its boundary when projected into $S \times 0$. With such a representative of the homotopy class of H^a , a handle Q has twists if and only if there is no homotopy of Q in $S \times \mathbb{R}$ relative to its boundary arcs q_1 and q_2 in $H(a)$ such that $\partial\pi(Q)$ is embedded in $S \times 0$.

Suppose that for arbitrarily small δ , $H^{a+\delta}$ is obtained from H^a by attaching a handle Q as described above. Suppose also that the projection of ∂H^a to $S \times 0$ only contains essential intersections. Let $h(t)$

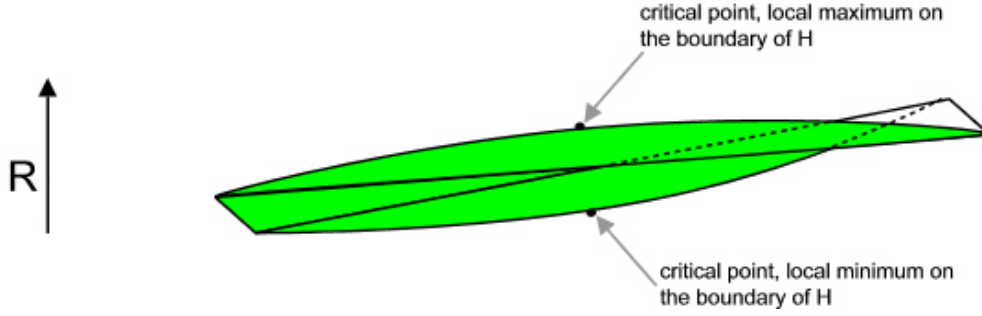


FIGURE 2. Cell decomposition of a bow tie handle. One side of the handle is shown in green.

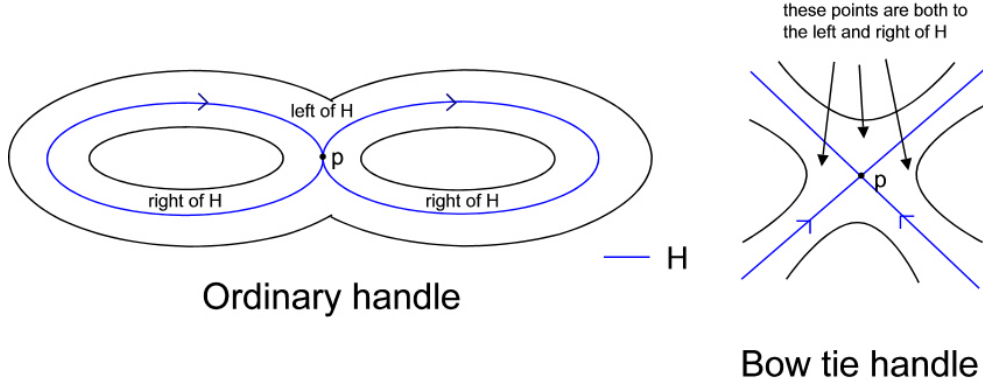
be a homotopy of $H^{a+\delta}$ in $S \times \mathbb{R}$ that fixes H^a . If for arbitrarily small δ there does not exist a $h(t)$ such that the image of Q under the homotopy $h(1)(H^{a+\delta})$ has a boundary that projects one to one into $S \times 0$, Q will be called a *bow tie handle*.

If Q is a bow tie handle that is attached to the surface F and $F \cup Q$ is homotopic to the surface F with an ordinary handle attached, Q will be called a *fake bow tie handle*. An example is given in figure 7.

The Morse function m_R will not give a handle decomposition that contains a bow tie handle. In other words, a bow tie handle is a union of 2-handles of the handle composition obtained from m_R , as shown in figure 2. As a result of lemma 3.1, it can be assumed without loss of generality (and will be) that no handle has more than one half twist.

4.3. The types of handles and classification of level sets. Since H is embedded, whenever a is not a critical value, $H(a) := H \cap (S \times a)$ is a union of curves and arcs that project one to one into $S \times 0$. If a is a critical value and $H_0^{a+\delta}$ is obtained from $H_0^{a-\delta}$ by attaching a handle, $H(a)$ is a one dimensional cell complex. The representative of the homotopy class of H was chosen such that there can be at most one critical point for any value of R , so δ can be chosen small enough to ensure that $H^{a+\delta}$ is obtained from $H^{a-\delta}$ either by gluing a 2-disc along a boundary component of $H_{a-\delta}$ or by attaching a single handle.

In the second case, there is a point p in $H_{a-\delta}^a$ such that $H_{a-\delta}^a$ has a component consisting of two 2-cells attached at a vertex p . If the handle is an ordinary handle, these two cells project onto two subsurfaces of $S \times 0$, both of which are either oriented as subsurfaces of $S \times 0$ or as subsurfaces of $-S \times 0$. If the handle is a bow tie handle, the two cells will have opposite orientations when projected onto subsurfaces of $S \times 0$.

FIGURE 3. Intersection of the I bundle of H with $S \times a$

By theorems 6.1 and 6.3 of [4], it is possible to find a smooth injective map ϕ from the normal bundle of $H_{a-\delta}^{a+\delta}$ into $S \times \mathbb{R}$ whose image, N , is embedded. As shown in the right half of figure 3, if $H(a)$ consists of two arcs crossing over at p , $\pi((S \times a) \cap (N \setminus H_{a-\delta}^{a+\delta}))$ is connected. Since H is embedded, and p is not allowed to be a degenerate critical point, the only possibility is that $H_{a-\delta}^{a+\delta}$ has a bow tie handle.

Whenever the handle is an ordinary handle, $\pi(H(a))$ is separating in $\pi((S \times a) \cap N)$. In this case, the point p is a point at which two arcs in $H(a)$ touch but do not cross over.

If a bow tie handle could have more than one half twist, there would be more than one point of self-intersection of $H(a)$, but not all of these self-intersections would be essential.

4.4. Proof of theorem in the absence of bow tie handles.

Proof of Theorem 1.1. It can be assumed without loss of generality that H is in general position. To start off with, suppose also that H is embedded.

Let c be a simple curve in the intersection of H with a level set of \mathbb{R} , $S \times a$. An *up collar* of c is an annular subsurface of H , $c \times [0, \epsilon]$, such that $c \times (0, \epsilon] \subset S \times (a, \infty)$ and $c \times [0, \epsilon]$ is not contained in a component of $H \cap S \times [a, \infty)$ consisting of an annulus with core curve c or a punctured sphere with boundary curves either contractible or homotopic to c . A *down collar* is defined analogously.

Let c_0 be a curve on ∂H . Since c_0 is simple, it is possible to assume without loss of generality that the zero of R was chosen to contain the boundary curve of H homotopic to c_0 .

Lemma 4.1 (Existence of an up/down collar). *Assume H has no bow tie handles. There exists an embedded representative of the homotopy class of H such that, by mapping R to $-R$ if necessary, the boundary curve c_0 has an up collar.*

Proof. H is orientable, so c_0 could not be the core curve of a Möbius band. Therefore c_0 is a boundary curve of an annulus A contained in H .

By cutting an annulus off one boundary component of H if necessary, it is possible to assume without loss of generality that c_0 is not on the boundary of a component of $H \cap (S \times [a, \infty))$ consisting of an annulus with core curve c_0 or a punctured sphere with boundary curves either contractible or homotopic to c_0 . (Since H is assumed to be incompressible, a curve on H that is contractible in $S \times \mathbb{R}$ is also contractible in H .)

All that could go wrong is therefore that the second boundary curve of A might have nonzero winding number in $S \times \mathbb{R}$ around c_0 . Let $\dot{c}_0(t)$ be a nonvanishing tangent vector to the curve c_0 , let $\dot{r}(t)$ be a nonvanishing vector field along c_0 tangent to A and linearly independent to $\dot{c}_0(t)$, and let n be a normal vector to $S \times a$. If the second boundary curve of A has nonzero winding number around c_0 , the handedness of $(\dot{c}_0(t), \dot{r}(t), n)$ has to change when moving around c_0 . Therefore, A has to be constructed by gluing together cells, some of which project to cells in $S \times 0$ with the induced subsurface orientation, and some of which project to cells in $S \times 0$ with the opposite of the induced subsurface orientation i.e. A has to contain bow tie handles. □

Remark 4.2. What the previous lemma does not show is that there is an embedded representative of the homotopy class containing H whose intersection with $S \times 0$ contains the multicurve m_1 such that every curve in m_1 simultaneously has an up collar.

4.5. Choosing the zero of \mathbb{R} . It is very convenient to choose $S \times 0$ such that the boundary curve c_0 is contained in $S \times 0$, however this choice results in a boundary curve c_0 consisting of degenerate critical points. This detail is resolved in the next lemma.

Lemma 4.3. *Suppose the zero of R was chosen such that the boundary curve c_0 of H is contained in $S \times 0$. Then there exists an embedded surface H_{wiggled} homotopic to H with the following properties:*

- (1) H_{wiggled} is arbitrarily close to H in the Hausdorff topology
- (2) the restriction of R to H_{wiggled} is a Morse function, and

- (3) *there exists a noncritical value r of R such that $S \times r$ intersects H_{wiggled} along the collar of c_0 whose existence was shown in the previous lemma such that H_{wiggled}^r contains a curve homotopic to c_0 .*

Proof. It follows from lemma 3.1 that there is a surface H_{wiggled} homotopic to H and arbitrarily close to H in the Hausdorff topology to which the restriction of the R coordinate is a Morse function. Whenever H_{wiggled} is sufficiently close to H in the Hausdorff topology, the boundary curve c_0 has to have a collar in H_{wiggled} such that the intersection of $S \times r$ with this collar contains a curve homotopic in H_{wiggled} to the boundary curve c_0 , for some small, noncritical value r of R . \square

It can therefore be assumed without loss of generality that the zero of the R coordinate and the embedded representative of the homotopy class of H are chosen such that the restriction of R to H is a Morse function, $H(0)$ contains a curve homotopic to the boundary curve c_0 and no two critical points occur at the same value of R . With this choice of the zero of the R coordinate, the boundary curves m_2 and $m_1 \setminus c_0$ might intersect $S \times 0$ in a complicated way.

5. HANDLE DECOMPOSITION

It is finally possible to start the handle decomposition. If a is so small that there are no critical points of R in the interval $[0, a]$, then H_0^a is a union of annuli whose core curves project onto a multicurve in S and perhaps some simply connected components.

Suppose now that a is large enough to ensure that there is only one critical value, b , in the interval $[0, a]$. If H_0^a contains a simply connected component that intersects some $S \times (a - \delta)$ along an arc or a contractible curve, and if this component wasn't in H_0^x for $x < b$, then the critical point has not changed the topology of the component of H_0^a with c_0 on its boundary.

Similarly, if one of the boundary components of H_0^a has a local minimum at $R = b$, this can only change the representatives of the free homotopy classes of the curves on the boundary of H_0^x as x passes through b , as shown in figure 4.

If a is large enough for there to be a saddle point p in the interior of H_0^a , this saddle point could cancel out a local minimum. A critical point of this type also only changes the representatives of the homotopy classes of the curves on the boundary of H_0^a and/or the number of contractible components.

If a is now chosen such that in the interval $(0, a]$ there is either:

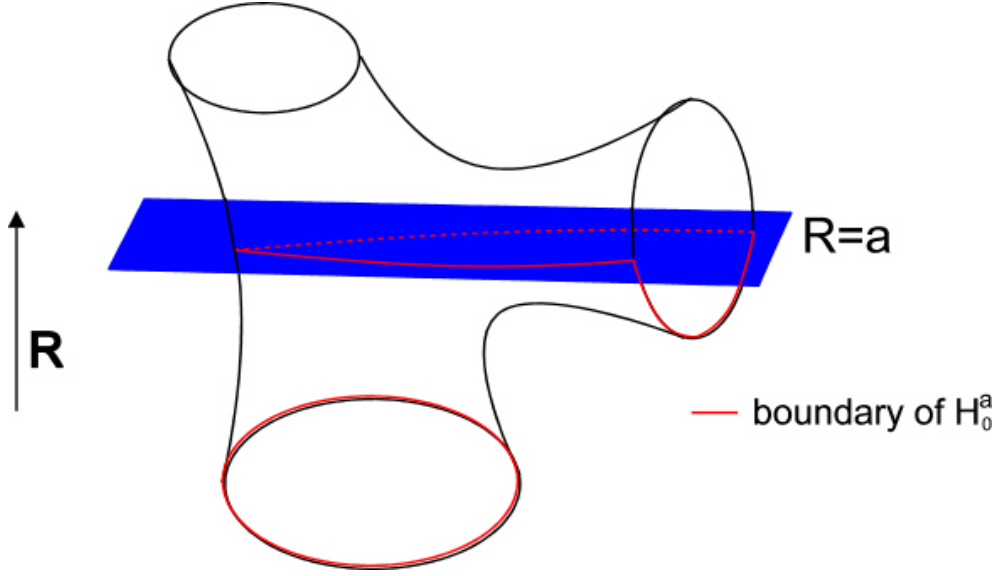


FIGURE 4. A local minimum on the boundary at $R = b$ for $b < a$.

- (1) a local maximum (either in the interior or on the boundary) or
- (2) a saddle point that does not cancel out a local minimum,

then the topology of H_0^x changes as x moves through the critical value b . In particular, H_0^a is obtained from H_0^b (a disjoint union of contractible components and annuli whose core curves project onto a multicurve in S) by attaching a handle. If this handle has one endpoint on a contractible component of H_0^b , again, the topology of the component to which the handle was added doesn't change when passing through the critical value. Otherwise, the endpoints of the handle are either both on the same annulus or on two different annuli. Whenever both of the endpoints of the handle are on the boundary of the annulus with core curve c_0 , H_0^a contains a pair of pants with boundary curves c_0 and $\eta \cup \beta$.

Remark. There is a second alternative here, namely that the handle has one endpoint on each boundary component of the annulus with core curve c_0 . However, this doesn't happen here, because the handle is attached to the boundary component $H(b)$ which only contains one curve homotopic to c_0 .

If neither η nor β is contractible, $\eta \cup \beta$ is homotopic to a multicurve because it is a submanifold of the intersection of the embedded surface H with $S \times b$, where the assumption that the handle is not a bow tie

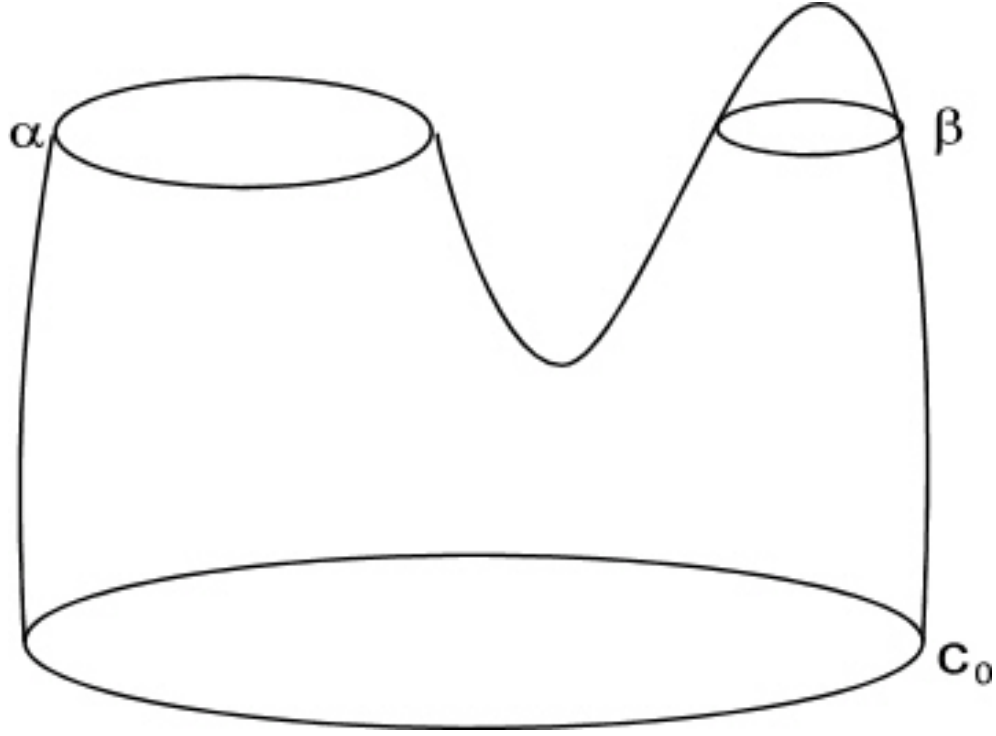


FIGURE 5. An annulus in H that intersects $S \times a$ in a disconnected set.

handle is being used here.

If none of the curves are contractible, $c_0 \cup \eta \cup \beta$ is also a multicurve, because $\eta \cup \beta$ is constructed by attaching a single handle to c_0 , where the handle is a subsurface of $S \times a$ without self intersections that meets the projection of c_0 onto $S \times a$ only at its endpoints (The assumption that the handle is not a bow tie handle is being used here also). Since $c_0 \cup \eta \cup \beta$ is a multicurve, the corresponding pair of pants projects onto a pair of pants in $S \times 0$. Let δ_1 be the multicurve $\eta \cup \beta$, unless one of η or β is contractible. If one of η or β is contractible, β for example, it follows from the assumption of incompressibility that H contains an annulus with boundary curves c_0 in $S \times 0$ and η in $S \times a$ that intersects $S \times i$ for some values of i in a disconnected set.

Similarly if one of the endpoints of the handle is on the boundary of the annulus with core curve c_0 and the other is on the boundary of another annulus whose core curve η is in the multicurve m_0 . In this case β is the curve obtained by connecting the annuli with core curves c_0 and η by a handle, and $c_0 \cup \eta \cup \beta$ is a multicurve for the same reason

as in the previous case. In this case, let δ_1 be the multicurve $\eta \cup \beta$ unless one of η or β is contractible.

If the handle doesn't have an endpoint on the annulus with core curve c_0 , then the intersection of H_0 with $S \times a$ will be a union of arcs plus a new multicurve, m_{0*} , containing c_0 . That m_{0*} is a multicurve follows from the assumption that there are no bow tie handles as before.

Following lemma 4.1 it can be assumed without loss of generality that the component of H_0 with the boundary curve c_0 of H does not consist of an annulus with core curve c_0 or a punctured sphere whose boundary curves are either contractible or homotopic to c_0 . Therefore, if a is increased enough, there will be a critical point of R on the component of H_0^a with c_0 on its boundary. Since there are only finitely many critical points, eventually the desired pair of pants is obtained, and δ_1 is then defined to be $\eta \cup \beta$.

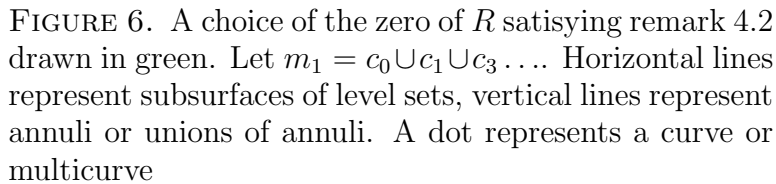
To construct δ_2 , cut the pair of pants with boundary $c_0 \cup \delta_1$ off H to obtain an embedded surface H_1 whose boundary contains the curves δ_1 . Since δ_1 is a multicurve, the previous argument can be applied with δ_1 in place of c_0 and H_1 in place of H , to show that there exists a union of multicurves, $\delta_1, \delta_2, \dots, \delta_n$ that decompose H into a union of subsurfaces. It follows from the definition of ordinary handle that each of these subsurfaces are homotopic to subsurfaces of $S \times 0$.

The proof is not finished yet, because it has not been shown that $m_1, \delta_1, \delta_2, \dots, \delta_n, m_2$ is a path in $\mathcal{HC}(S, \alpha)$. However, as illustrated in figure 6, the decomposition of H by the multicurves $\delta_1, \delta_2, \dots, \delta_n$ makes it possible to find an explicit choice of the zero of R such that remark 4.2 is satisfied. With this choice of zero of R , all the curves in m_1 have up collars, and $H(0)$ contains m_1 . The previous argument then gives $\gamma_1 := (m_1 \cup \eta \cup \beta) \setminus c_0$ if the handle has both endpoints on the annulus with core curve c_0 or $\gamma_1 := (m_1 \cup \beta) \setminus (c_0 \cup \eta)$ otherwise. This completes the proof of theorem 1.1 in the absence of bow tie handles.

Remark - immersed surfaces. The previous paragraph highlights the point at which the proof of theorem 1.1 breaks down for immersed surfaces. A weaker statement is obtained in lemma 5.1.

5.1. Proof with bow tie handles. It remains to prove the theorem in the case of bow tie handles.

Examples of orientable surfaces in $S \times \mathbb{R}$ with boundary $m_2 - m_1$, whose handle decomposition contains bow tie handles are not difficult to construct. For example, given a simple path m_1, γ_1, m_2 , construct a surface in $S \times \mathbb{R}$ by gluing together two subsurfaces that project onto subsurfaces of $S \times 0$; one with the subsurface orientation of $S \times 0$ and one with the opposite orientation.



The key observation here is that the resulting bow tie handles occur in pairs, otherwise the boundary of the surface could not be a union of two multicurves. Similarly, when constructing paths in $\mathcal{HC}(S, \alpha)$, points of intersection were removed in pairs, where each pair consisted of intersections with opposite handedness. When attempting to construct a pants decomposition of H corresponding to a path $m_1, \gamma_1, \gamma_2, \dots, m_2$ in $\mathcal{HC}(S, \alpha)$, no γ_i is not allowed to separate a bow tie handle in H from its partner. The argument in this section shows

that this restriction is sufficient to obtain a path $m_1, \gamma_1, \gamma_2, \dots, m_2$ in $\mathcal{HC}(S, \alpha)$ from which H is constructed.

Suppose β_i is a union of (not necessarily simple) curves on H such that

- $H \setminus \beta_i$ has two components; one with boundary curves m_1 and the other with boundary curves m_2
- $\beta_i - m_1$ is the boundary of a subsurface of H whose handle decomposition by m_R consists of i handles, which are not allowed to be bow tie handles unless they are fake
- i is as large as possible.

Note that it does not follow from the previous arguments that β_i is a multicurve, because it has not been shown that it is contained in any level set of the R coordinate.

Remark. The union of curves, β_i , is not necessarily unique. Suppose for example that for $h < i$, the subsurface H^a has a curve c in β_h on its boundary. Suppose also that $H^{a+\delta}$ is constructed by attaching a bow tie handle to the boundary component c , and for some $\iota > \delta$, $H^{a+\iota}$ is constructed from $H^{a+\delta}$ by attaching an ordinary handle. There can be some choices involved in how the ordinary handle is attached to the surface if the bow tie handle has not been attached first. We simply make some choice and do not worry about uniqueness.

If β_{i+1} is constructed from β_i by attaching a bow tie handle with a half twist, β_{i+1} could only have smaller self-intersection number if the subsurface of H with bow tie handle attached is homotopic to the same subsurface with an ordinary handle attached as shown in figure 7. If the bow tie handle attached to H_0^a to obtain $H_0^{a+\delta}$ is not fake, then $\pi\partial H_0^{a+\delta}$ is a one dimensional cell complex with a subcomplex homotopic in $S \times \mathbb{R}$ to $\pi\partial H_0^a$. In this case, the intersection number of $\partial H_0^{a+\delta}$ with m_2 or with itself can not have been decreased by attaching the bow tie handle. It follows that the handle decomposition of H can not consist of bow tie handles only.

Since the handle decomposition of the subsurface of H with boundary $m_2 - \beta_i$ consists of bow tie handles only, β_i can not intersect m_2 . Also, m_2 is a multicurve, so β_i can not have self intersections, since these can not be removed by attaching bow tie handles. Therefore $m_2 - \beta_i$ is a multicurve. The handle decomposition of the subsurface of H bounded by $m_2 - \beta_i$ can not consist of bow tie handles only, so it must be the case that β_i is homotopic to m_2 . The theorem then follows from the

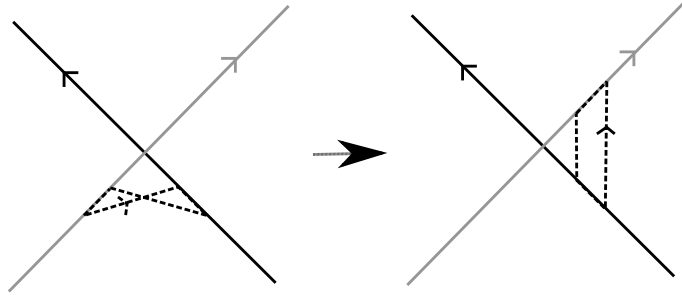


FIGURE 7. A fake bow tie handle is shown on the left. This handle is homotopic to the ordinary handle shown on the right.

proof of the result under the assumption that there are no bow tie handles. □

5.2. Immersed Surfaces. One reason for studying $\mathcal{HC}(S, \alpha)$ is that it can be used for constructing minimal genus surfaces in $S \times \mathbb{R}$. Not all such minimal genus surfaces are embedded, as shown in [6] example 13. For immersed surfaces in $S \times \mathbb{R}$, a slightly weaker form of theorem 1.1 is obtained.

Lemma 5.1. *Suppose H is an oriented, immersed, incompressible surface in $S \times \mathbb{R}$. Then H is homotopic to a union of subsurfaces T_i glued along homotopic boundary curves, as described in section 2. Each of the T_i is homotopic to an embedded subsurface of $S \times 0$.*

Proof. If H is not embedded, it follows from standard arguments based on theorem 3.3 of [8] that there is a finite index covering space, $\tilde{S} \times \mathbb{R}$, of $S \times \mathbb{R}$, such that H can be lifted to an embedded surface, \tilde{H} , in $\tilde{S} \times \mathbb{R}$. Let $\tilde{\alpha}$ be the integral homology class of \tilde{S} containing the lift of m_1 to the boundary of \tilde{H} .

The same arguments as in the proof of theorem 1.1 show that \tilde{H} is homotopic to a surface constructed from a path $\tilde{m}_1, \tilde{\delta}_1, \tilde{\delta}_2, \dots, \tilde{m}_2$ as described in section 2. The surface \tilde{S} can be covered by a finite number of neighbourhoods that project one to one onto S , and so therefore can \tilde{H} , from which the lemma follows. □

Note that it is not possible to argue, as in the proof of theorem 1.1, that the decomposition of H into the subsurfaces T_i determines a path in $\mathcal{HC}(S, \alpha)$. All that can be shown is that each curve in $\tilde{\delta}_i$ projects onto a curve in S , and that the projection of ∂T_i is a multicurve. It

does not follow that the projection of $\tilde{\delta}_i$ is a multicurve.

REFERENCES

- [1] I. Agol, J. Hass, and W. Thurston. The computational complexity of knot genus and spanning area. *Transactions of the American Mathematical Society*, 358, 2006.
- [2] M. Bestvina, K. Bux, and D. Margalit. The dimension of the Torelli group. *J. Amer. Math. Soc.*, 23:61–105, 2010.
- [3] A. Hatcher. The cyclic cycle complex of a surface. arXiv:0806.0326v1, 2008.
- [4] M. Hirsch. *Differential Topology*. Springer, 1997.
- [5] I. Irmer. *The curve graph and surface construction in $S \times \mathbb{R}$* . PhD thesis, Universität Bonn, July 2010.
- [6] I. Irmer. Geometry of the Homology Curve Complex. arXiv:1107.3547, 2011.
- [7] J. Milnor. *Lectures on the h-cobordism Theorem*. Princeton University Press, 1965.
- [8] P. Scott. Subgroups of surface groups are almost geometric. *J. London Math. Soc.*, 1978.

E-mail address: irmer@cs.uni-bonn.de